Minimizing Inequity in Facility Location Games

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Introduction

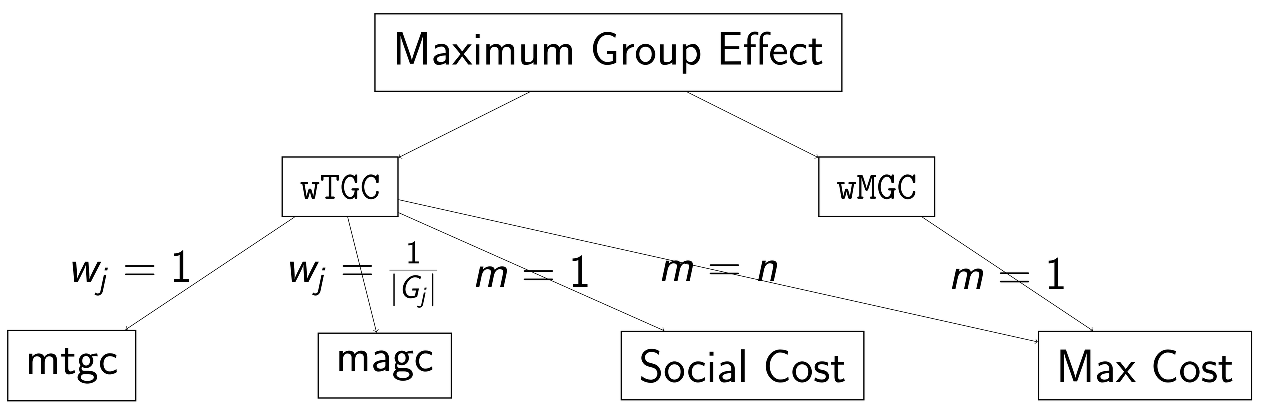
We study facility location on the real line with agents who may belong to one or multiple groups. Marsh and Schilling (1994) introduced a center objective that minimizes the maximum group burden, but much of the subsequent literature effectively collapses this goal to minimizing the maximum individual distance, thereby overlooking group-level inequities. We return to the original group-centric perspective by formalizing group effects through total or maximum (weighted) distances, and we develop strategyproof mechanisms that provide meaningful fairness guarantees at the group level. In doing so, we resolve a central open question posed by Zhou, Li, and Chan (2022).

Model

- A set of n agents $N = \{1, 2, \cdots, n\}$.
- A set of m groups $\mathcal{G} = \{G_1, G_2, \cdots, G_m\}$.
- Each group G_j has a weight w_j. Let $w_{\max} = \max_{j\in[m]} w_j$, $w_{\min} = \min_{j\in[m]} w_j$.
- Each agent i's profile $\theta = (x_i, g_i)$ where $x_i \in \mathbb{R}$ is the *private* location and $g_i \subseteq \mathcal{G}$ is the membership of groups.
- The cost incurred by agent i is defined as $c_i(f(\boldsymbol{\theta}), x_i) = \min_{y\in f(\theta)} |y - x_i|$.
- Maximum Group Effect: $\text{mge}(\boldsymbol{\theta}, f(\boldsymbol{\theta})) = \max_{j\in[m]} E_j$, where
  $E_j = w_j \cdot \Sigma_{i\in G_j} c(f(\boldsymbol{\theta}), x_i)$ (weighted Total Group Cost); or
  $E_j = w_j \cdot \max_{i\in G_j} c(f(\boldsymbol{\theta}), x_i)$ (weighted Maximum Group Cost).
- Strategyproofness: A mechanism f is strategyproof if, for any agent i with true location x_i and group g_i, any misreported location $x_i' \in \mathbb{R}$, and any profile $\boldsymbol{\theta}'_{-i}$ of other agents' reports, we have:
  $$c(f((x_i, g_i), \boldsymbol{\theta}'_{-i}), x_i) \leq c(f((x_i', g_i), \boldsymbol{\theta}'_{-i}), x_i).$$
- Approximation Ratio: For any mechanism f, the approximation ratio is:
  $$\rho = \sup_{\boldsymbol{\theta}\in\Theta^n} \frac{\text{mge}(\boldsymbol{\theta}, f(\boldsymbol{\theta}))}{\text{mge}(\boldsymbol{\theta}, \text{OPT}(\boldsymbol{\theta}))},$$
  where $\text{OPT}(\boldsymbol{\theta})$ is the optimal facility placement that minimizes mge objective under the profile $\boldsymbol{\theta}$.

Our Results

Generalization of Objective Metric



Technical Results

(All listed bounds are tight. Gray shading denotes our contributions.)

Setting	Objectives	Mechanisms	Bounds
$k = 1$	wTGC	BALANCED	2
	wMGC	MAJOR-PHANTOM	2
$k = 2$	wTGC	ENDPOINT	$1 + (n-2)\frac{w_{\max}}{w_{\min}}$
	wMGC	ENDPOINT	$1 + \frac{w_{\max}}{w_{\min}}$
$k \geq 3$	General	/	∞

Comparative Analysis ($k = 1$)

Objectives	Mechanisms	Bounds
wTGC	MEDIAN	$1 + \frac{w_{\max}}{w_{\min}}(m-1)$
	LEFTMOST	$1 + \frac{w_{\max}}{w_{\min}}(n-1)$
	MGDM	$1 + 2\frac{w_{\max}}{w_{\min}}$
	BALANCED	**2**
wMGC	MEDIAN	$1 + \frac{w_{\max}}{w_{\min}}$
	LEFTMOST	
	MGDM	
	MAJOR-PHANTOM	**2**

Single-Facility Scenario

weighted Total Group Cost (wTGC)

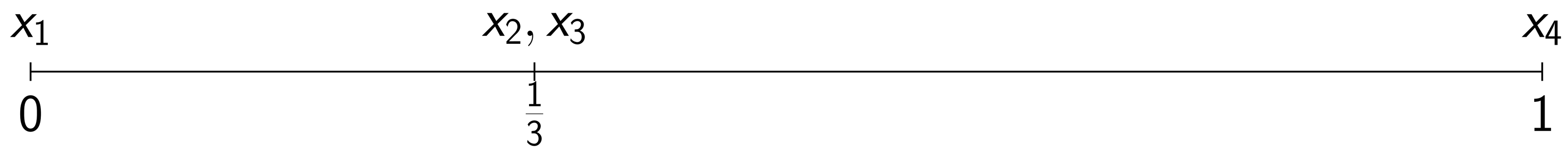
BALANCED Mechanisms

Input: Agent profile $\boldsymbol{\theta}$, group weights $\{w_j\}_{j\in[m]}$.

1. Define $L_j(y) \leftarrow |\{i \in N : x_i \leq y \text{ and } j \in g_i\}|$ and $R_j(y) \leftarrow |\{i \in N : x_i > y \text{ and } j \in g_i\}|$.
2. Compute
$$f(\boldsymbol{\theta}) \leftarrow \min\left\{y \in \mathbb{R} : \max_{j\in[m]} w_j L_j(y) \geq \max_{j'\in[m]} w_{j'} R_{j'}(y)\right\}.$$

Output: Facility location $f(\boldsymbol{\theta})$.

Example: 4 agents: $N = \{1, 2, 3, 4\}$. 3 groups $\mathcal{G} = \{g_1, g_2, g_3\}$. Weights: $w_1 = w_2 = w_3 = 1$. Group membership: $g_1 = \{1, 3\}$, $g_2 = \{2\}$, $g_3 = \{2, 3\}$, $g_4 = \{3\}$.



Condition	Group 1	Group 2	Group 3
$y \leq 0$	$L_1(y) = 1$ $R_1(y) = 0$	$L_2(y) = 0$ $R_2(y) = 2$	$L_3(y) = 1$ $R_3(y) = 2$
$0 < y \leq \frac{1}{3}$	$L_1(y) = 1$ $R_1(y) = 0$	$L_2(y) = 2$ $R_2(y) = 0$	$L_3(y) = 2$ $R_3(y) = 1$
$\frac{1}{3} < y \leq 1$	$L_1(y) = 1$ $R_1(y) = 0$	$L_2(y) = 2$ $R_2(y) = 0$	$L_3(y) = 3$ $R_3(y) = 0$

Facility Location $f(\theta) = \frac{1}{3}$.

weighted Maximum Group Cost (wMGC)

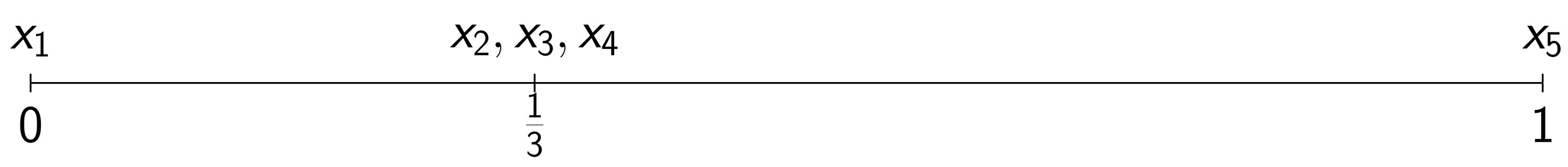
MAJOR-PHANTOM Mechanisms

Input: Agent profile $\boldsymbol{\theta}$, group weights $\{w_j\}_{j\in[m]}$.

1. Let $G_{\max}$ denote the largest weight group and $\mathbf{x}^{G_{\max}} = \{x_1^{G_{\max}}, \ldots, x_{|G_{\max}|}^{G_{\max}}\}$ denote the location profile of agents in G^*, tie-breaking in favor of the smallest index.
2. Let $v_1 \leq \cdots \leq v_{|G_{\max}|-1}$ denote $|G_{\max}| - 1$ values $v_1 \leq \cdots \leq v_{|G_{\max}|-1}$.
3. $f(\boldsymbol{\theta}) \leftarrow \text{median}(\mathbf{x}^{G_{\max}}, v_1, \ldots, v_{|G_{\max}|-1})$, tie-breaking by selecting the leftmost.

Output: Facility location $f(\boldsymbol{\theta})$.

Example: 5 agents: $N = \{1, 2, 3, 4, 5\}$. 3 groups $\mathcal{G} = \{g_1, g_2, g_3\}$. Weights: $w_1 = 1, w_2 = w_3 = 2$. Group membership: $g_1 = \{1, 2\}$, $g_2 = \{2\}$, $g_3 = \{2, 3\}$, $g_4 = \{2, 3\}$, $g_5 = \{1, 3\}$.



- First consider $G_{\max} = G_2$ where $\mathbf{x}^{G_{\max}} = \{x_1, x_2, x_3, x_4\}$.
- Next consider 3 values $v_1 = v_2 = v_3 = -\infty$.
- Let $f(\boldsymbol{\theta}) \leftarrow \text{median}(x_1, x_2, x_3, x_4, v_1, v_2, v_3) = 0$.
- Output the facility location $f(\boldsymbol{\theta}) = 0$.

Facility Location $f(\boldsymbol{\theta}) = 0$.

Future Works

- Randomized mechanisms for the **maximum group effect** objective.
- Adapting the **maximum group effect** objective to higher dimensional metric space.

Reference

Marsh, M. T.; and Schilling, D. A. 1994. Equity measurement in facility location analysis: A review and framework. *European journal of operational research*, 74(1): 1–17.

Zhou, H.; Li, M.; and Chan, H. 2022. Strategyproof Mechanisms for Group-Fair Facility Location Problems. In *Proceedings of the 31st International Joint Conference on Artificial Intelligence and the 25th European Conference on Artificial Intelligence (IJCAI)*, 613–619.